

Solution of constrained mechanical multibody systems using Adomian decomposition method

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Abstract. Constrained mechanical multibody systems arise in many important applications like robotics, vehicle and machinery dynamics and biomechanics of locomotion of humans. These systems are described by the Euler-Lagrange equations which are index-three differential-algebraic equations (DAEs) and hence difficult to treat numerically.

The purpose of this paper is to propose a novel technique to solve the Euler-Lagrange equations efficiently. This technique applies the Adomian decomposition method (ADM) directly to these equations. The great advantage of our technique is that it neither applies complex transformations to the equations nor uses index-reductions to obtain the solution. Furthermore, it requires solving only linear algebraic systems with a constant nonsingular coefficient matrix at each iteration. The technique developed leads to a simple general algorithm that can be programmed in Maple or Mathematica to simulate real application problems. To illustrate the effectiveness of the proposed technique and its advantages, we apply it to solve an example of the Euler-Lagrange equations that describes a two-link planar robotic system.

Keywords: Euler-Lagrange equations, multibody systems; differential-algebraic equations; Adomian decomposition method

1 Introduction

Constrained mechanical multibody systems arise in many areas of applications such as robotics, biomechanics of locomotion of humans and dynamics of vehicle and machinery [1, 2, 3, 4]. The

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dynamical behavior of constrained multibody systems is described by the Euler-Lagrange equations

$$\begin{cases} \frac{dp}{dt} = v, \\ M(p) \frac{dv}{dt} = f(p, v) - G^T(p) \lambda, \\ 0 = g(p), \quad t \geq 0. \end{cases} \quad (1)$$

Here t is the time, $p(t) \in \mathbb{R}^{n_p}$ and $v(t) \in \mathbb{R}^{n_p}$ specify the positions and the orientations of all bodies and their velocities, respectively. The vector $\lambda \in \mathbb{R}^{n_\lambda}$ is the vector of Lagrange multipliers. The matrix $M(p) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p}$ is a mass matrix. The mapping $f : \mathbb{R}^{n_p} \times \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_p}$ defines the applied and internal forces (other than the constraint forces), whereas $g : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_\lambda}$, ($n_\lambda \leq n_p$) defines the constraints. The term $G(p)^T \lambda$ represents the constraint forces, where $G(p) = \partial g / \partial p \in \mathbb{R}^{n_\lambda \times n_p}$ denotes the Jacobian of $g(p)$.

The Euler-Lagrange equations (1) form a nonlinear system of differential-algebraic equations (DAEs). Consistent initial conditions

$$p(0) = p_0, \quad v(0) = v_0, \quad (2)$$

are necessary to uniquely determine a solution. The given vectors p_0 and v_0 , which specify the initial configuration and initial velocity, are chosen so that the consistency equations

$$g(p_0) = 0, \quad (3)$$

$$G(p_0)v_0 = 0, \quad (4)$$

are satisfied. For the variable $\lambda(t)$, no initial condition is prescribed because $\lambda(0)$ is already determined by DAE (1) and initial conditions (2).

Throughout this paper, we assume that $M(p)$, $f(p, v)$ and $g(p)$ are analytical. A standard assumption on the Jacobian $G(p)$ is the full row rank condition

$$\text{rank } G(p) = n_\lambda, \quad (5)$$

which means that the constraint equations defined by $g(p)$ are linearly independent.

In addition the matrix M is assumed to be symmetric and positive definite, that is

$$z^T M(p) z > 0, \text{ for all } z \in \text{Ker } G(p). \quad (6)$$

If assumptions (5)-(6) hold, then the matrix

$$\begin{pmatrix} M(p) & G^T(p) \\ G(p) & 0 \end{pmatrix}, \quad (7)$$

is nonsingular and DAE (1) is therefore index-three. For any consistent initial conditions (2), system (1) has a unique solution [1]. The index we consider here is the differential index which is the minimum number of times that all or a part of the DAE must be differentiated with respect to time in order to obtain an ordinary differential equation [5]. It is well-known that index-three DAEs present difficulties for numerical integration methods [6]. Therefore, several techniques have been proposed in the literature to solve DAE system (1) [6, 7, 8, 9, 12, 13]. A very popular way is to reduce the index by differentiating the constraints one or more times with respect to time before applying numerical integration methods. However, the main problem with this technique is that the numerical solution of the index-reduced system may no longer satisfy the constraints of original DAE (1) due to error propagation. Constraints violation, known also as drift-off phenomena, leads to non physical solutions. To overcome this difficulty, some techniques like stabilization or augmented Lagrangian formulation have been proposed to keep the constraint violations under control during the numerical integration [7, 8, 9, 10]. The most popular stabilization method is that of Baumgarte [7], but its drawback is the way of choosing its feedback parameters. The augmented Lagrangian formulation [11] has the same problem of parameter selection. The challenge is therefore to construct efficient methods that provide solutions of the Euler-Lagrange equations which satisfy the constraints in these equations.

The Adomian decomposition method (ADM) and its modifications [15, 16, 17, 18, 19, 20, 21] are known to be efficient methods in solving a large variety of linear and nonlinear problems in science and engineering. Among these problems, we mention algebraic equations [15], ordinary differential equations [16, 17, 18, 19, 20, 21], partial differential equations [22] and integral equations [23].

In this work, we present a new approach to solve the Euler-Lagrange equations using ADM. The solution by this method satisfies all the DAE constraints. The ADM is first applied directly to the Euler-Lagrange equations where the nonlinear terms are expanded using the Adomian polynomials [24, 25, 26, 27, 28, 29]. Based on the index of the Euler-Lagrange equations, a nonsingular linear algebraic recursion system is derived for the expansion components of the solution. Our technique has the great advantage that it does not use complex transformations like index reductions before applying the ADM to the equations. To demonstrate the effectiveness of the proposed method, we solve an example of the Euler-Lagrange equations that models a two-link planar robotic system. Further, our technique is based on a simple algorithm that can be programmed in Maple or Mathematica to simulate real application problems.

This paper is organized as follows: in section 2, we review the ADM for solving ordinary differential equations. Next, in section 3, we present our method for the solution of the Euler-Lagrange equations. Then, in section 4, we apply the developed technique to solve an example of the Euler-Lagrange equations that models a two-link planar robotic system. Finally, a discussion and a conclusion are given in sections 5 and 6, respectively.

2 Adomian decomposition method

In this section, we give a brief review for the Adomian decomposition method (ADM) [15, 16, 17, 18, 19, 20, 21] to solve ordinary differential equations. For this purpose, let us consider the following nonlinear differential equation

$$Lu + Ru + N(u) = f, \quad (8)$$

where L is an easily invertible operator (usually taken as the highest-order derivative), R is an operator grouping the remaining lower-order derivatives, $N(u)$ is the nonlinear term and f is a given analytical function.

Solving equation (8) for Lu then applying the inverse operator L^{-1} to both sides, we obtain

$$L^{-1}Lu = L^{-1}f - L^{-1}Ru - L^{-1}N(u). \quad (9)$$

If $Lu = du/dt$ and the initial condition $u(t_0) = u_0$ is given, then L^{-1} represents the integral from t_0 to t and $L^{-1}Lu = u - u_0$.

$$u = u_0 + L^{-1}f - L^{-1}Ru - L^{-1}N(u). \quad (10)$$

To apply the ADM to equation (10), we first assume that the solution u of (8) to have the infinite series form

$$u = \sum_{n=0}^{\infty} u^{(n)}, \quad (11)$$

where the unknown solution components $u^{(n)}$, $n = 0, 1, 2, \dots$ are to be determined later by the method.

Second, the nonlinear term $N(u)$ is expanded in an infinite series in terms of the Adomian polynomials $N^{(n)}$ [24, 25, 26, 27, 28, 29] as

$$N(u) = \sum_{n=0}^{\infty} N^{(n)}(u^{(0)}, \dots, u^{(n)}). \quad (12)$$

Substituting (11) and (12) into (10) and choosing $u^{(0)}$ as

$$u^{(0)} = u_0 + L^{-1}f, \quad (13)$$

we obtain

$$\sum_{n=0}^{\infty} u^{(n)} = u^{(0)} - L^{-1}R \sum_{n=0}^{\infty} u^{(n)} - L^{-1} \sum_{n=0}^{\infty} N^{(n)}. \quad (14)$$

Comparing the general term on the left hand side with that on the right hand side, we derive the following recursion scheme for the ADM

$$u^{(n)} = -L^{-1}Ru^{(n-1)} - L^{-1}N^{(n-1)}, \quad n \geq 1. \quad (15)$$

Since $u^{(0)}$ is known, recursion (15) can be used to generate as many solution components $u^{(n)}$ as one wants. Further, if series (11) converges then it gives the exact solution of (8) and an approximation of order n_0 to solution can be obtained from

$$u = \sum_{n=0}^{n_0-1} u^{(n)}. \quad (16)$$

To compute the Adomian polynomials $N^{(n)}$, $n = 0, 1, \dots$ associated with the nonlinearity $N(u)$, one can use the following definition for all forms of nonlinearity

$$N^{(n)} := N^{(n)}(u^{(0)}, \dots, u^{(n)}) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left(N \left(\sum_{i=0}^{\infty} \lambda^i u^{(i)} \right) \right)_{\lambda=0}, \quad n \geq 0. \quad (17)$$

Using this formula, we obtain the following first few Adomian polynomials

$$\begin{aligned} N^{(0)} &= N(u^{(0)}), \\ N^{(1)} &= u^{(1)} N'(u^{(0)}), \\ N^{(2)} &= u^{(2)} N'(u^{(0)}) + \frac{(u^{(1)})^2}{2!} N''(u^{(0)}), \\ N^{(3)} &= u^{(3)} N'(u^{(0)}) + u^{(1)} u^{(2)} N''(u^{(0)}) + \frac{(u^{(1)})^3}{3!} N'''(u^{(0)}), \\ N^{(4)} &= u^{(4)} N'(u^{(0)}) + \left(\frac{(u^{(2)})^2}{2!} + u^{(1)} u^{(3)} \right) N''(u^{(0)}) + \frac{(u^{(1)})^2 u^{(2)}}{2!} N'''(u^{(0)}) + \frac{(u^{(1)})^4}{4!} N''''(u^{(0)}), \end{aligned} \quad (18)$$

where the dash (') represents the differentiation with respect to u .

In a similar manner, one can easily generate the remaining polynomials from (17). In the literature, there are several algorithms for computing the Adomian polynomials without the need for formula (17), but a more convenient algorithm for the m -variable case is recently proposed in [29]

$$N^{(n)} = \frac{1}{n} \sum_{k=1}^m \sum_{i=0}^{n-1} (i+1) v_k^{(i+1)} \frac{\partial N^{(n-1-i)}}{\partial v_k^{(0)}}, \quad n \geq 1. \quad (19)$$

3 The proposed method

In this section, we present our method for solving the Euler-Lagrange equations. These equations are known to be difficult to treat numerically since they represent an index-three system of differential-algebraic equations (DAEs). The technique we propose here is based on the Adomian decomposition method (ADM). To solve these equations, we first apply the ADM directly to them and expand the nonlinear terms using the Adomian polynomials. Then, an algebraic linear recursion system for the solution expansion components is derived. Taking account of the index of the DAE, this algebraic system is shown to be uniquely solvable for the solution expansion components. The main advantage of our technique is that it does not require to transform the equations to lower index DAEs before applying the ADM to them. We start our technique by the following proposition.

Proposition: Let $u = (u_1, \dots, u_l)^\top \in \mathbb{R}^l$, $v = (v_1, \dots, v_n)^\top \in \mathbb{R}^n$ be two vectors, $A := (A(u)_{i,j}) \in \mathbb{R}^m \times \mathbb{R}^n$ be a matrix, $u = \sum_{k=0}^\infty u^{(k)}$, $v = \sum_{k=0}^\infty v^{(k)}$, $u^{(k)} = (u_1^{(k)}, \dots, u_l^{(k)})$, $v^{(k)} = (v_1^{(k)}, \dots, v_n^{(k)})$. Let $A^{(k)} = (A_{i,j}^{(k)})$, where $A_{i,j}^{(k)} = A_{i,j}^{(k)}(u^{(0)}, \dots, u^{(k)})$ denotes the k -th Adomian polynomial of the entry $A_{i,j}$. Then, given A and v , the Adomian polynomials of $z = Av$ and $z = A^{-1}v$ are given by

- (a) $z^{(k)} = \sum_{l=0}^k A^{(l)} v^{(k-l)} = \sum_{l=0}^k A^{(k-l)} v^{(l)}$ and
- (b) $A^{(0)} z^{(k)} = v^{(k)} - \sum_{l=0}^{k-1} A^{(k-l)} z^{(l)}$, $k = 0, 1, \dots$

Proof:

- (a) Let $z = Av$, then the Adomian polynomials $z^{(k)}$ of z are given by

$$z^{(k)} = (Av)^{(k)} = \begin{pmatrix} \left(\sum_{j=1}^n A_{1,j} v_j \right)^{(k)} \\ \vdots \\ \left(\sum_{j=1}^n A_{m,j} v_j \right)^{(k)} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n (A_{1,j} v_j)^{(k)} \\ \vdots \\ \sum_{j=1}^n (A_{m,j} v_j)^{(k)} \end{pmatrix}, \quad (20)$$

$$\begin{aligned} &= \begin{pmatrix} \sum_{j=1}^n \sum_{l=0}^k A_{1,j}^{(k-l)} v_j^{(l)} \\ \vdots \\ \sum_{j=1}^n \sum_{l=0}^k A_{m,j}^{(k-l)} v_j^{(l)} \end{pmatrix} = \begin{pmatrix} \sum_{l=0}^k \sum_{j=1}^n A_{1,j}^{(k-l)} v_j^{(l)} \\ \vdots \\ \sum_{l=0}^k \sum_{j=1}^n A_{m,j}^{(k-l)} v_j^{(l)} \end{pmatrix}, \quad (21) \\ &= \sum_{l=0}^k A^{(k-l)} v^{(l)}. \end{aligned}$$

- (b) Let $Az = v$ then, using (a), the Adomian polynomials $v^{(k)}$ of v are given by

$$v^{(k)} = (Az)^{(k)} = \sum_{l=0}^k A^{(k-l)} z^{(l)} = A^{(0)} z^{(k)} + \sum_{l=0}^{k-1} A^{(k-l)} z^{(l)}.$$

The Adomian polynomials $z^{(k)}$ of $z = A^{-1}v$ are then given by

$$A^{(0)}z^{(k)} = v^{(k)} - \sum_{l=0}^{k-1} A^{(k-l)}z^{(l)}.$$

Now to solve DAE system (1), let $Lp(t) = dp/dt$ and $L^{-2}p(t) = L^{-1}(L^{-1}p(t))$. Then $L^{-1}p(t) = \int_0^t p(t) dt$ and $L^{-2}p(t) = \int_0^t \int_0^t p(t) dt dt$.

We solve the second equation of (1) for dv/dt as

$$\frac{dv}{dt} = M^{-1}(p) \left(f(p, v) - G^T(p) \lambda \right). \quad (22)$$

Applying the operator L^{-1} to both sides of the first equation of (1) and (22) then using initial conditions (2), we get

$$\begin{cases} p = p_0 + L^{-1}v, \\ v = v_0 + L^{-1} \left(M^{-1}(p) \left(f(p, v) - G^T(p) \lambda \right) \right). \end{cases} \quad (23)$$

Then, we expand the solution components p, v and λ as

$$p = \sum_{n=0}^{\infty} p^{(n)}, \quad v = \sum_{n=0}^{\infty} v^{(n)}, \quad \lambda = \sum_{n=0}^{\infty} \lambda^{(n)}, \quad (24)$$

where the unknowns $p^{(n)}, v^{(n)}$ and $\lambda^{(n)}$, $n = 0, 1, 2, \dots$ will be determined later by our method. The nonlinear terms $f(p, v)$ and $g(p)$ are also expanded in infinite series as

$$f(p, v) = \sum_{n=0}^{\infty} f^{(n)}, \quad g(p) = \sum_{n=0}^{\infty} g^{(n)}, \quad (25)$$

where $f^{(n)} := f^{(n)}(p^{(0)}, v^{(0)}, \dots, p^{(n)}, v^{(n)})$ and $g^{(n)} := g^{(n)}(p^{(0)}, \dots, p^{(n)})$ denote the Adomian polynomials. Using (19), the Adomian polynomials $f^{(n)}$ and $g^{(n)}$, $n = 0, 1, 2, \dots$ can be written as

$$f^{(n)} = \begin{cases} f(p^{(0)}, v^{(0)}), & n = 0, \\ \frac{1}{n} \sum_{i=1}^{n-1} i \left(\frac{\partial f^{(n-i)}}{\partial p^{(0)}} \right) p^{(i)} + i \left(\frac{\partial f^{(n-i)}}{\partial v^{(0)}} \right) v^{(i)} + \left(\frac{\partial f^{(0)}}{\partial p^{(0)}} \right) p^{(n)} + \left(\frac{\partial f^{(0)}}{\partial v^{(0)}} \right) v^{(n)}, & n \geq 1, \end{cases} \quad (26)$$

and

$$g^{(n)} = \begin{cases} g(p^{(0)}), & n = 0, \\ \frac{1}{n} \sum_{i=1}^{n-1} i \left(\frac{\partial g^{(n-i)}}{\partial p^{(0)}} \right) p^{(i)} + \left(\frac{\partial g^{(0)}}{\partial p^{(0)}} \right) p^{(n)}, & n \geq 1. \end{cases} \quad (27)$$

Substituting expansions (24) into equations (23) and the third equation of (1), we get

$$\begin{cases} \sum_{n=0}^{\infty} p^{(n)} = p_0 + \sum_{n=0}^{\infty} L^{-1} v^{(n)}, \\ \sum_{n=0}^{\infty} v^{(n)} = v_0 + \sum_{n=0}^{\infty} L^{-1} \left(M^{-1}(p) \left(f(u, v) - G^T(p) \lambda \right) \right)^{(n)}, \\ 0 = \sum_{n=0}^{\infty} g^{(n)}. \end{cases} \quad (28)$$

Choosing the initial terms $p^{(0)}$ and $v^{(0)}$ as

$$p^{(0)} = p_0, \quad v^{(0)} = v_0, \quad (29)$$

then comparing the general terms on the left and right hand sides of (28), we obtain the following recursion system

$$\begin{cases} p^{(n)} = L^{-1} v^{(n-1)}, \\ v^{(n)} = L^{-1} \left(M^{-1}(p) \left(f(p, v) - G^T(p) \lambda \right) \right)^{(n-1)}, \\ 0 = g^{(n)}, \quad n \geq 1. \end{cases} \quad (30)$$

This system leads to the following recursion system

$$\begin{cases} p^{(n)} = L^{-2} \left(M^{-1}(p) \left(f(p, v) - G^T(p) \lambda \right) \right)^{(n-2)}, \\ 0 = g^{(n)}, \quad n \geq 2, \end{cases} \quad (31)$$

where

$$v^{(n-1)} = L p^{(n)}. \quad (32)$$

Using expansions (25) and the previous proposition, we calculate the right hand side of the first equation of system (31), and obtain the following linear algebraic system for the unknowns $p^{(n)}$ and $L^{-2} \lambda^{(n-2)}$

$$\begin{aligned} M(p^{(0)}) p^{(n)} + G^T(p^{(0)}) L^{-2} \lambda^{(n-2)} &= R, \\ G(p^{(0)}) p^{(n)} &= S, \quad n \geq 2, \end{aligned} \quad (33)$$

where

$$R = L^{-2} \left(f^{(n-2)} - \sum_{k=0}^{n-3} \left(\left(G^T(p) \right)^{(n-2-k)} \lambda^{(k)} + (M(p))^{(n-2-k)} z^{(k)} \right) \right),$$

and

$$S = -\frac{1}{n} \sum_{i=1}^{n-1} i \left(\frac{\partial g^{(n-i)}}{\partial p^{(0)}} \right) p^{(i)}.$$

The iterates $z^{(k)}$, $k = 2, \dots, n-3$ are computed from

$$M(p^{(0)}) z^{(k)} = f^{(k)} - G^\top(p^{(0)}) \lambda^{(k)} - \sum_{l=0}^{k-1} \left((G^\top(p))^{(k-l)} \lambda^{(l)} + (M(p))^{(k-l)} z^{(l)} \right). \quad (34)$$

In system (33), the right hand side depends only on previous iterations $p^{(n-1)}, \dots, p^{(0)}, \lambda^{(n-3)}, \dots, \lambda^{(0)}$ and $v^{(n-1)}, \dots, v^{(0)}$. Since the Jacobian G has full row rank and the matrix M is positive definite, then system (33) determines $p^{(n)}$ and $L^{-2}\lambda^{(n-2)}$ uniquely for $n \geq 2$.

One way to solve system (33) is to multiply the first equation of this system from left by the matrix $G(p^{(0)})M^{-1}(p^{(0)})$. Then substitute $G(p^{(0)})p^{(n)}$ by its expression from the second equation of (33), to obtain the following nonsingular algebraic system for the unknown $L^{-2}\lambda^{(n-2)}$

$$G(p^{(0)})M^{-1}(p^{(0)})G^\top(p^{(0)})L^{-2}\lambda^{(n-2)} = G(p^{(0)})M^{-1}(p^{(0)})R - S. \quad (35)$$

Since rank condition (5) holds and M is positive definite, equation (35) can be solved uniquely for $L^{-2}\lambda^{(n-2)}$ to get

$$L^{-2}\lambda^{(n-2)} = \left(G(p^{(0)})M^{-1}(p^{(0)})G^\top(p^{(0)}) \right)^{-1} (G(p^{(0)})M^{-1}(p^{(0)})R - S). \quad (36)$$

Now applying the operator L^2 to both sides of equation (36), we can determine the unknown $\lambda^{(n-2)}$

$$\lambda^{(n-2)} = \left(G(p^{(0)})M^{-1}(p^{(0)})G^\top(p^{(0)}) \right)^{-1} (G(p^{(0)})M^{-1}(p^{(0)})L^2R - L^2S). \quad (37)$$

Then, substituting the expression of $L^{-2}\lambda^{(n-2)}$ into the first equation of (33), we determine the unknown $p^{(n)}$

$$p^{(n)} = M^{-1}(p^{(0)})R - M^{-1}(p^{(0)})G^\top(p^{(0)}) \left(G(p^{(0)})M^{-1}(p^{(0)})G^\top(p^{(0)}) \right)^{-1} (G(p^{(0)})M^{-1}(p^{(0)})R - S). \quad (38)$$

Now, using equation (32) we can calculate $v^{(n-1)}$. Finally, we obtain an approximate solution for DAE initial-value problem (1)-(2) as

$$p(t) = \sum_{n=0}^{n_0-1} p^{(n)}, \quad v(t) = \sum_{n=0}^{n_0-2} v^{(n)}, \quad \lambda(t) = \sum_{n=0}^{n_0-3} \lambda^{(n)}, \quad (39)$$

where n_0 is the order of approximation of $p(t)$.

4 Application

In this section, we illustrate and demonstrate the effectiveness of our technique to solve Euler-Lagrange equations (1)-(2) which describe the motion of constrained mechanical multibody systems. These equations are known to be difficult to solve numerically because they are index-three differential-algebraic equations (DAEs). Following the procedure developed in the previous section, we first apply the Adomian decomposition method (ADM) directly to these equations without using complex transformations like index-reductions. Then, we expand the nonlinear terms using the Adomian polynomials. Taking account of the index-three condition, we derive a nonsingular linear algebraic recursion system for the expansion components of the solution. Finally, by solving this algebraic system, we obtain the solution of the Euler-Lagrange equations. As a test problem, we consider the following example of constrained multibody system made up from example 6.4 in [12] which describes a two-link planar robotic system, where the mass matrix is

$$M(\theta_1, \theta_2) = \begin{pmatrix} m_1 l_1^2/3 + m_2 (l_1^2 + l_2^2/3 + l_1 l_2 \cos \theta_2) & m_2 (l_2^2/3 + (1/2) l_1 l_2 \cos \theta_2) \\ m_2 (l_2^2/3 + (1/2) l_1 l_2 \cos \theta_2) & m_2 l_2^2/3 \end{pmatrix}.$$

The force term is

$$f(\theta_1, \theta_2, d\theta_1/dt, d\theta_2/dt) = \begin{pmatrix} \left(l_1 \cos \theta_1 + l_2 \cos (\theta_1 + \theta_2) \right) (d\theta_1/dt) - 3\theta_1 \\ \left(l_2 \cos (\theta_1 + \theta_2) \right) (d\theta_1/dt) + (1 - (3/2) \cos \theta_2) \theta_1 \end{pmatrix},$$

and the constraint function is given by

$$g(\theta_1, \theta_2) = l_1 \sin \theta_1 + l_2 \sin (\theta_1 + \theta_2).$$

Taking $l_1 = l_2 = 1, m_1 = m_2 = 3$ and using the notation of (1), with $n_p = 2, n_\lambda = 1, p = (p_1, p_2)^\top = (\theta_1, \theta_2)^\top, v = dp/dt$, we have

$$M(p) = \begin{pmatrix} 5 + 3 \cos p_2 & (3/2) \cos p_2 \\ 1 + (3/2) \cos p_2 & 1 \end{pmatrix},$$

and

$$f(p, v) = \begin{pmatrix} \left(\cos p_1 + \cos (p_1 + p_2) \right) v_1 - 3p_1 \\ \left(\cos (p_1 + p_2) \right) v_1 + (1 - (3/2) \cos p_2) p_1 \end{pmatrix}.$$

The constraint function becomes

$$g(p) = \sin p_1 + \sin (p_1 + p_2),$$

and its Jacobian $G(p) = (\cos p_1 + \cos(p_1 + p_2), \cos(p_1 + p_2))$ is full row rank $n_\lambda = 1$.

For the consistent initial conditions

$$p(0) = (0, 0)^\top, \quad v(0) = (1, -2)^\top, \quad (40)$$

the exact solution for this example is $p(t) = (\sin t, -2 \sin t)^\top$, $v(t) = (\cos t, -2 \cos t)^\top$ and $\lambda(t) = \cos t$.

The Euler-Lagrange equations corresponding to this example form an index-three DAE and therefore difficult to solve numerically. Using the procedure developed in the previous section, system (33) can be solved for $n = 2, 3, \dots$ to reveal the dynamics of the mechanical system.

For $n = 2$, we get

$$p^{(2)} + M^{-1}(p^{(0)}) G^\top(p^{(0)}) L^{-2} \lambda^{(0)} = L^{-2} M^{-1}(p^{(0)}) f^{(0)}, \quad (41)$$

$$G(p^{(0)}) p^{(2)} = 0. \quad (42)$$

Now, since

$$M^{-1}(p^{(0)}) = (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix}, \quad G(p^{(0)}) = (2, 1),$$

$$p^{(0)} = \begin{pmatrix} p_1^{(0)} \\ p_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad v^{(0)} = \begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

and

$$f^{(0)} = \begin{pmatrix} (\cos p_1^{(0)} + \cos(p_1^{(0)} + p_2^{(0)})) v_1^{(0)} - 3p_1^{(0)} \\ (\cos(p_1^{(0)} + p_2^{(0)})) v_1^{(0)} + (1 - (3/2) \cos p_2^{(0)}) p_1^{(0)} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

equations (41)-(42) reduce to

$$p^{(2)} + (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} \lambda^{(0)} = (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} (1), \quad (43)$$

$$(2, 1) p^{(2)} = 0. \quad (44)$$

Multiplying system (43) from left by the matrix $(2, 1)$ then using (44), we get

$$(2/7) (2, 1) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} \lambda^{(0)} = (2/7) (2, 1) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} (1), \quad (45)$$

which leads

$$\lambda^{(0)} = 1. \quad (46)$$

Now substituting the value of $\lambda^{(0)}$ from (46) into (43), we obtain

$$p^{(2)} + (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2}(1) = (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2}(1), \quad (47)$$

which gives

$$p^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (48)$$

and using (32), we have

$$v^{(1)} = Lp^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (49)$$

For $n = 3$, we have

$$\begin{aligned} p^{(3)} + M^{-1}(p^{(0)}) G^T(p^{(0)}) L^{-2} \lambda^{(1)} &= L^{-2} M^{-1}(p^{(0)}) \left(f^{(1)} \right. \\ &\quad \left. - \left(G^T(p) \right)^{(1)} \lambda^{(0)} - (M(p))^{(1)} z^{(0)} \right), \\ G(p^{(0)}) p^{(3)} &= 0. \end{aligned} \quad (50)$$

Now, since

$$(M(p))^{(1)} = \begin{pmatrix} -3p_2^{(1)} \sin p_2^{(0)} & -(3/2) p_2^{(1)} \sin p_2^{(0)} \\ -(3/2) p_2^{(1)} \sin p_2^{(0)} & 0 \end{pmatrix} = 0, \quad \left(G^T(p) \right)^{(1)} = 0,$$

and

$$f^{(1)} = \begin{pmatrix} -3 \\ -1/2 \end{pmatrix} t,$$

system (50) reduces to

$$p^{(3)} + (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} \lambda^{(1)} = (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} -3 \\ -1/2 \end{pmatrix} L^{-2}(t), \quad (51)$$

$$(2, 1) p^{(3)} = 0. \quad (52)$$

Multiplying system (51) from left by the matrix $(2, 1)$ then using (52), we get

$$\begin{aligned} (2/7) (2, 1) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} \lambda^{(1)} &= (2/7) (2, 1) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} -3 \\ -1/2 \end{pmatrix} L^{-2}(t), \\ &= 0, \end{aligned} \quad (53)$$

which gives

$$\lambda^{(1)} = 0. \quad (54)$$

Now substituting the value of $\lambda^{(1)}$ from (54) into (51), we obtain

$$p^{(3)} = (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} -3 \\ -1/2 \end{pmatrix} L^{-2}(t), \quad (55)$$

which gives

$$p^{(3)} = -\frac{t^3}{3!} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad (56)$$

and

$$v^{(2)} = Lp^{(3)} = -\frac{t^2}{2!} \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \quad (57)$$

For $n = 4$, we have

$$\begin{aligned} p^{(4)} + M^{-1}(p^{(0)}) G^T(p^{(0)}) L^{-2} \lambda^{(2)} &= L^{-2} M^{-1}(p^{(0)}) \left(f^{(2)} \right. \\ &\quad \left. - \left(G^T(p) \right)^{(2)} \lambda^{(0)} - (M(p))^{(2)} z^{(0)} \right. \\ &\quad \left. - \left(G^T(p) \right)^{(1)} \lambda^{(1)} - (M(p))^{(1)} z^{(1)} \right), \end{aligned} \quad (58)$$

$$G(p^{(0)})p^{(4)} = -\frac{1}{4} \sum_{i=1}^3 i \left(\frac{\partial g^{(4-i)}}{\partial p^{(0)}} \right) p^{(i)},$$

and

$$f^{(2)} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} t^2, \quad \left(G^T(p) \right)^{(2)} = \begin{pmatrix} -1 \\ -1/2 \end{pmatrix} t^2.$$

The iterates $z^{(0)}$ is calculated from

$$M(p^{(0)}) z^{(0)} = f^{(0)} - G^T(p^{(0)}) \lambda^{(0)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} (1) = 0,$$

which gives $z^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus (58) becomes

$$p^{(4)} + (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} \lambda^{(2)} = (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} (-1/2 t^2), \quad (59)$$

$$(2, 1)p^{(4)} = 0. \quad (60)$$

Multiplying system (59) from left by the matrix $(2, 1)$ then using (60), we get

$$(2/7) (2, 1) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} \lambda^{(2)} = (2/7) (2, 1) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} (-1/2t^2), \quad (61)$$

which gives

$$\lambda^{(2)} = -1/2t^2. \quad (62)$$

Now substituting the value of $\lambda^{(2)}$ from (62) into (59), we obtain

$$p^{(4)} + (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} (-1/2t^2) = (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} (-1/2t^2),$$

which gives

$$p^{(4)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (63)$$

and using (32), we have

$$v^{(3)} = Lp^{(4)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (64)$$

For $n = 5$, we have

$$\begin{aligned} p^{(5)} + M^{-1} (p^{(0)}) G^T (p^{(0)}) L^{-2} \lambda^{(3)} &= L^{-2} M^{-1} (p^{(0)}) \left(f^{(3)} \right. \\ &\quad \left. - \sum_{k=0}^2 \left(\left(G^T (p) \right)^{(3-k)} \lambda^{(k)} + (M(p))^{(3-k)} z^{(k)} \right) \right), \quad (65) \\ G(p^{(0)}) p^{(5)} &= -\frac{1}{5} \sum_{i=1}^4 i \left(\frac{\partial g^{(5-i)}}{\partial p^{(0)}} \right) p^{(i)}, \end{aligned}$$

where

$$\begin{aligned} f^{(3)} &= \begin{pmatrix} 1/2 \\ 37/12 \end{pmatrix} t^3, \quad \left(G^T (p) \right)^{(3)} = 0. \\ (M(p))^{(2)} &= \begin{pmatrix} -6t^2 & -3t^2 \\ -3t^2 & 0 \end{pmatrix}. \end{aligned}$$

The term $z^{(1)}$ is calculated from

$$\begin{aligned} M(p^{(0)}) z^{(1)} &= f^{(1)} - \left(G^T (p) \right)^{(1)} \lambda^{(0)} - G^T (p^{(0)}) \lambda^{(1)} - (M(p))^{(1)} z^{(0)} \\ &= f^{(1)} = \begin{pmatrix} -3 \\ -1/2 \end{pmatrix} t, \end{aligned} \quad (66)$$

which gives $z^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} t$. Thus system (65) becomes

$$p^{(5)} + (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} \lambda^{(3)} = (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/12 \end{pmatrix} L^{-2} (t^3), \quad (67)$$

$$(2, 1)p^{(5)} = 0. \quad (68)$$

Multiplying system (67) from left by the matrix $(2, 1)$ then using (68), we get

$$(2/7)(2, 1) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} L^{-2} \lambda^{(3)} = (2/7)(2, 1) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/12 \end{pmatrix} L^{-2} (t^3), \\ = 0, \quad (69)$$

which gives

$$\lambda^{(3)} = 0. \quad (70)$$

Now substituting the value of $\lambda^{(3)}$ from (70) into (67), we obtain

$$p^{(5)} = (2/7) \begin{pmatrix} 2 & -5 \\ -5 & 16 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/12 \end{pmatrix} L^{-2} (t^3), \quad (71)$$

which gives

$$p^{(5)} = \frac{t^5}{5!} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad (72)$$

and using (32), we have

$$v^{(4)} = Lp^{(5)} = \frac{t^4}{4!} \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \quad (73)$$

Continuing this process until $n = 8$, we obtain the following ADM solution

$$p(t) = t \begin{pmatrix} 1 \\ -2 \end{pmatrix} - \frac{t^3}{3!} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{t^5}{5!} \begin{pmatrix} 1 \\ -2 \end{pmatrix} - \frac{t^7}{7!} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad (74)$$

$$v(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} - \frac{t^2}{2!} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{t^4}{4!} \begin{pmatrix} 1 \\ -2 \end{pmatrix} - \frac{t^6}{6!} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad (75)$$

and

$$\lambda(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}. \quad (76)$$

These are the first few terms of Taylor expansions, around $t = 0$, of

$$p(t) = (\sin t, -2 \sin t)^\top, \quad v(t) = (\cos t, -2 \cos t)^\top, \quad \lambda(t) = \cos t, \quad (77)$$

which is the exact solution of the problem in this example.

5 Discussion

The Euler-Lagrange equations are known to be difficult to solve numerically. The reason is that they form an index-three system of differential-algebraic equations (DAEs). In this paper, we propose a novel technique that applies the Adomian decomposition method (ADM) directly to solve the Euler-Lagrange equations. This technique has successfully handled these equations without the need for complex transformations like index-reductions. This method transforms these equations into easily solvable algebraic systems for the expansion components of the solution. To illustrate the effectiveness of the proposed technique, an example of the Euler-Lagrange equations describing a two-link planar robot system is solved. This example shows that the ADM is a simple powerful tool to obtain the exact or approximate solutions of the Euler-Lagrange equations.

6 Conclusion

This work presents the analytical solution of the Euler-Lagrange equations using the ADM. A procedure for solving these is presented. The technique was tested on an example of the Euler-Lagrange equations that describes a two-link robot system. The results obtained show that the proposed method can be applied to solve the Euler-Lagrange equations efficiently to obtain the exact or an approximate solution. On the one hand, it is important to note that these types of equations are difficult to solve and on the other, the direct application of the ADM was able to solve the Euler-Lagrange equations. Also, it is important to note that, our technique does not make transformations to the equations before applying the ADM to them. The technique is based on a straightforward procedure that can be programmed in Maple or Mathematica to simulate real application problems. Finally, further work is needed to apply a multistage ADM form to solve the Euler-Lagrange equations and other semi-explicit nonlinear higher-index DAEs.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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